Forum for Linguistic Studies 2024, 6(2),
https://doi.org/10.24294/fls.v6i2.2135
ISSN:2705-0610 E-ISSN:2705-0602

t-CF PEBBLING NUMBER OF SOME STANDARD GRAPHS

C. Muthulakshmi@sasikala[1], A. Arul Steffi[2]

[1] Department of Mathematics, Sri Paramakalyani College, Alwarkurichi, 627412,
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012,
Tamil Nadu, India.
kalasasispkc@gmail.com.

[2]Research Scholar [Reg. No. 20121282092007], Department of Mathematics,
St. Xavier's College (Autonomous), Palayamkottai, 627002,
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012,
Tamil Nadu, India.
email: arulsteffi96@gmail.com

Abstract

Graph pebbling is a mathematical process involving the movement of pebbles on a connected graph G according to specific rules. Assume G is a graph with some pebbles distributed over its vertices. A CF pebbling move is defined as the removal of x pebbles from one vertex, followed by discarding $\lfloor \frac{x}{2} \rfloor$ pebbles and moving the remaining $\lceil \frac{x}{2} \rceil$ pebbles to an adjacent vertex. The t-CF pebbling number, $\lambda_t(G)$, of a connected graph G, is the least positive integer n such that any distribution of n pebbles on G allows t pebbles to be carried to any arbitrary vertex using a sequence of CF pebbling moves. In this study, we determine the t-CF pebbling number of path graph, star graph, complete graph and fan graph.

Keywords: CF pebbling move, t-CF pebbling number, path, star, fan graph.

1. Introduction

Pebbling in Graphs was first studied by Chung[1]. Pebbling numbers are a central concept in graph theory, focusing on the movement and placement of resources (pebbles) on graph vertices under specific rules. The t-ceiling floor (t-CF) pebbling number is a variation that imposes unique constraints, combining aspects of resource management with vertex-specific requirements.

The t-CF pebbling number of a graph G, denoted as $\lambda_t(G)$, is the minimum number of pebbles required to ensure that t pebbles can be placed on a specified target vertex, starting from any initial pebble configuration, while satisfying the ceiling floor constraints. These constraints involve specific rules about the number of pebbles distributed on vertices, often guided by rounding-up (ceiling) and rounding-down (floor) functions.

In this paper, we study the t-CF pebbling number of path graph, star graph complete graph and fan graph, analyzing how this number evolves with varying graph configurations. We present precise formulations for the t-CF pebbling number of path P_n of length n. Here, p(v) denotes the number of pebbles placed on the vertex v in a graph G and $t > 1$.

2. Preliminaries

Definition 2.1. [2] Assume G is a graph with some pebbles distributed over its vertices. A CF pebbling move is when x pebbles are removed from one vertex, $\lfloor \frac{x}{2} \rfloor$ pebbles are thrown away, and $\lceil \frac{x}{2} \rceil$ pebbles are moved to an adjacent vertex.

Definition 2.2. [2] A CF pebbling number $\lambda(G, v)$ of a vertex v of a graph G is the smallest number $\lambda(G, v)$ such that at least one pebble may be moved to the target vertex v using a sequence of CF pebbling moves, for any placement of $\lambda(G, v)$ pebbles on the vertices of G. The maximum $\lambda(G, v)$ over all the vertices of G is the CF pebbling number of a graph, denoted as $\lambda(G)$.

3. The t-CF pebbling number of some standard graphs

Definition 3.1. Let G be a graph with some pebbles distributed over its vertices. The t-CF pebbling number $\lambda_t(G)$ of a connected graph G is the least positive integer n such that any distribution of n pebbles on G allows t pebbles to be carried to any arbitrary vertex using a sequence of CF pebbling moves.

Theorem 3.1. For any graph G, $\lambda(G) \geq |V(G)|$.

Proof. Let G be a connected graph on $|V(G)|$ vertices. Let $r \in V(G)$ be the target vertex. If we place zero pebbles on r and place a single pebble on every vertex of $G\backslash\{r\}$, we cannot move a pebble to the target vertex r. So $\lambda(G) \geq |V(G)|$.

Theorem 3.2. If G contains a cut-vertex then $\lambda(G) \geq n$ where $n = |V(G)|$.

Proof. Let u be a cut vertex of G and let G_1 and G_2 be two distinct components of $G\backslash\{u\}$. Let $r \in G_1$ be the target vertex and let $y \in G_2$ be any vertex. By placing two pebbles in y, and placing a single pebble on each of the vertices of $G\backslash\{y, r, u\}$, we cannot move a pebble to r. So $\lambda(G) \geq n$.

Theorem 3.3. The 2-CF pebbling number of path P_n of length n is $2^n + 1$.

Proof. $2^n + 1$ can be divided into $2.2^{n-1} + 1 = 2^{n-1} + 2^{n-1} + 1$, using atmost 2^{n-1} pebbles, a pebble can be moved to any target vertex $v_i (1 \leq i \leq n + 1)$ of P_n, and using $2^{n-1} + 1$ pebbles, since CF pebbling number of path P_n of length n is $2^{n-1} + 1$, a pebble can be moved to any target vertex $v_i (1 \leq i \leq n + 1)$ of P_n.

If we place 2^n pebbles at v_1, then 2 pebbles cannot be moved to v_{n+1}. So $\lambda_2(P_n) \geq 2^n + 1$.

Theorem 3.4. For a path of length n, the t-CF pebbling number is $\lambda_t(P_n) = 1 + 2^n(t-1) \forall t \geq 2, n \geq 1$.

Proof. Let us assume v_{n+1} is our target vertex and $p(v_{n+1}) = 0$, and place $2^n(t-1)$ pebbles at v_1, we cannot move t pebbles to v_{n+1}, so $\lambda_t(P_n) \geq 1 + 2^n(t - 1)$.

When t=2, using $2^n + 1$ pebbles, two pebbles can be moved to any vertex $v_i (1 \leq i \leq n + 1)$ of P_n by theorem 3.3.

Assume $t > 2$, there exists atleast $2^{n+1} + 1$ pebbles, distributing $2^{n+1} + 1$ pebbles on all the vertices of P_n expect the target vertex $v_i (1 \leq i \leq n + 1)$ and as $p(v_i) = 0$, now $2^{n+1} + 1 = 2.2^n + 1 = 2^n + 2^n + 1$, using atmost 2^n, pebbles (since $d(v_i, v_j) \leq n \forall i \neq j$), a pebble can be moved to any target vertex, use the remaining $2^n(t - 1) + 1 - 2^n = 2^n(t - 2) + 1$ pebbles, t-1 pebbles can be moved to any target vertex by induction.

If $p(v_i) = l$ where $1 \leq l \leq t - 1$, then the total number of pebbles distributed in P_n except v_i is $1 + 2^n(t - 1) - l$ which is greater than $2^n(t - l - 1) + 1$ pebbles, we can move t-l pebbles to any
target vertex v_i by induction.

Definition 3.2. [3] A fan graph $F_n, n \geq 2$ is obtained by joining all vertices of a path P_n to a further vertex, called center $F_n = K_1 + P_n$, where $V(G) = n+2$, $E(G) = 2n+1$ and P_n is a path of length n.

Theorem 3.5. The 2-CF pebbling number $\lambda_2(P_n + K_1) = n + 4 \forall n \geq 2$.

Proof. Let $V(P_n + K_1) = \{v_0, v_1, \cdots, v_{n+1}\}$ and
$E(P_n + K_1) = \{v_0v_1, v_0v_2, \cdots, v_0v_{n+1}, v_1v_2, v_2v_3, \cdots, v_nv_{n+1}\}$. If we place four pebbles on v_{n+1}
and a single pebble on each of the vertices of $(P_n + K_1)\backslash\{v_0, v_1, v_{n+1}\}$, two pebbles cannot be moved to v_1. So $\lambda_2(P_n + K_1) > 4 + (n - 1) = n + 3$.

Assume v_0 is our target vertex and place zero pebbles on v_0. Distributing $n + 4$ pebbles on all the vertices of P_n, there exists a vertex with atleast three pebbles or there exist two vertices, with atleast two pebbles each and hence two pebbles could be moved to v_0.
Assume any vertex $v_i(1 \leq i \leq n + 1)$, be our target vertex. If $p(v_0) \geq 3$, then two pebbles could be moved to v_i. If $p(v_0) = 1$ or 2, there exists a vertex v_j, $j \neq 0$, i, receives atleast three pebbles (or) there exist three vertices each with two pebbles each, then we could move two pebbles to $v_i(1 \leq i \leq n + 1)$. If $p(v_0) = 0$, then for any distribution of n+2 pebbles a pebble could be moved to any vertex of $P_n + K_1$ in particular to v_i or there exist v_j with atleast five pebbles or
there exist three vertices with atleast two pebbles each and we are done.

Theorem 3.6. The t-CF pebbling number of $P_n + K_1$ is $\lambda_t(P_n + K_1) = 4(t-1) + n - 1$, $t > 1$.

Proof. Consider a distribution such that $p(v_1) = 4t - 4$, $p(v_0) = 0$, $p(v_j) = 1$, $p(v_n) = 0 = p(v_{n+1}) \forall j \neq 1, 0, n, n+1$ then t pebbles cannot be placed on v_1. $\lambda_t(P_n+K_1) \geq 4t-4+n-2+1 = 4(t - 1) + (n - 1)$.
When t=2, result is true by theorem 3.5. When $t > 2$, there are atleast n+7 pebbles on all the vertices of $P_n + K_1$. Using atmost eight pebbles two pebbles can be moved to the target vertex $v_i(i = 0, 1, 2, ...n)$. Use the remaining $4(t-1)+(n-1)-8 = 4(t-1-2)+(n-1) = 4(t-3)+(n-1)$. By induction, t-2 pebbles can be moved to our target vertex.

Theorem 3.7. The 2-CF pebbling number of $K_{1,n}$ is $\lambda_2(K_{1,n}) = n + 3$ for $n > 1$

Proof. Let $V(K_{1,n}) = A \cup B$ where $A = \{a_1\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

Claim: $\lambda_2(a_1, K_{1,n}) = n + 2$.

By placing n+2 pebbles on all the vertices of $K_{1,n}\backslash\{a_1\}$, there exists two vertices with atleast two pebbles or there exist a vertex with three pebbles, so two pebbles can be moved to a_1.

Claim: $\lambda_2(b_1, K_{1,n}) = n + 3$.

By placing $n + 3$ pebbles on a vertices of $K_{1,n}\backslash\{b_1\}$, consider the following cases.
If $P(a_1) \geq 3$, then two pebbles could be moved to b_1 from a_1. If $p(a_1) \leq 2$, then $K_{1,n}\backslash\{a_1, b_1\}$ has atleast n+1 pebbles. There exists atleast two vertices of B with two pebbles, so two pebbles
could be moved to b_1 if $p(a_1) = 1$ or 2.

If $p(a_1) = 0$, then all the $n + 3$ pebbles are placed on $K_{1,n}\backslash\{a_1, b_1\}$ then there exist atleast three vertices with two pebbles each or there exist atleast two vertices with three pebbles each or there exist a vertex with five pebbles, (otherwise number of pebbles placed on $K_{1,n}\backslash\{a_1, b_1\}$ is n+2, which is a contradiction, hence two pebbles could be moved to b_1.
Thus, $\lambda_2(b_1, K_{1,n}) = n + 3$. So, $\lambda_2(K_{1,n}) \leq n + 3$.
By placing four pebbles on b_n and placing a single pebble on each of the vertices of $K_{1,n}\backslash\{a_1, b_1, b_n\}$, we cannot move two pebbles to b_1, so, $\lambda_2(K_{1,n}) > n + 2$. Hence $\lambda_2(K_{1,n}) = n + 3$.

Theorem 3.8. The t-CF pebbling number of $K_{1,n}$ is $\lambda_t(K_{1,n}) = 4t + n - 5$ for $n, t > 1$. **Proof.** Let us prove the theorem by induction on t.

Claim: $\lambda_t(a_1, K_{1,n}) = 2t + n - 2$

By placing zero pebbles on a_1 and place $2t - 2$ pebbles on any vertex of B say b_1 and place a pebble at every vertices of $K_{1,n}\backslash\{a_1, b_1\}$, then t pebbles cannot be moved to a_1.
So $\lambda_t(a_1, K_{1,n}) > 2t + n - 3$.
When t =2, the result is true by theorem 3.7. Assume $t > 2$. The graph has atleast n+4 pebbles. Using atmost four pebbles two pebbles could be moved to a_1 by induction, the remaining $2t + n - 2 - 4 = 2(t - 2) + n - 2$ pebbles are used to move t-2 pebbles to a_1.
So, $\lambda_t(a_1, K_{1,n}) \leq 2t+n-2$.

Claim: $\lambda_t(b_1, K_{1,n}) = 4t + n - 5$

Assume $p(b_1) = 0$. If we place 4(t-1) pebbles on any of the vertices of B say b_n and place a single pebble on each of the vertices of $K_{1,n}\backslash\{a_1, b_1, b_n\}$, we cannot move t pebbles to b_1.
So $\lambda_t(b_1, K_{1,n}) \geq 4t + n - 5$.
When t =2, the result is true by theorem .7. Assume $t > 2$, the graph has atleast n+7 pebbles, using atmost eight pebbles, two pebbles could be moved to b_1. From the remaining $4t + n - 5 - 8 = 4(t - 2) + n - 5$, $t - 2$ pebbles could be moved to b_1, by induction. Hence $\lambda_t(b_1, K_{1,n}) \leq 4t + n - 5$. So $\lambda_t(K_{1,n}) = 4t + n - 5$.

Theorem 3.9. The 2-CF pebbling number of a complete graph K_n with n vertices is $n + 1$.

Proof. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Let $v \in V(K_n)$ be the target vertex. Suppose n pebbles are placed on the vertices of K_n as follows.

1. $p(w) = 2$ for some $w \neq v \in V(K_n)$.
2. $p(u) = 2 \;\; \forall \;\; u \neq v, w$ such that $u, v, w \in V(K_n)$

Clearly, two pebbles cannot be moved to v. Thus, $\lambda_2(K_n) \geq n + 1$.
Now, assume that $n + 1$ pebbles are placed on the vertices of K_n. Then the pebbles will be distributed in any one of the following ways.

1. There exists atleast one vertex u such that $p(u) \geq 3$.
2. There exists atleast two vertices u, w such that $p(u) \geq 2$ and $p(w) \geq 2$.

If both cases did not occur, then the number of pebbles placed in K_n is less than $n + 1$, which leads to a contradiction. Thus, in both ways, two pebbles can be moved to the target vertex v. Hence $\lambda_2(K_n) \leq n + 1$.

Theorem 3.10. The t-CF pebbling number of a complete graph K_n with n vertices is $n+2t-3$.

Proof. Let us prove the result by induction on n. The result is true for $t = 2$ according to theorem 3.9. Assume that the result is true for all values less than t. Now, assume that $n + 2t - 3$ pebbles are placed on the graph. Using $n + 2t - 5$ pebbles, $t - 1$ pebbles can be moved to the target vertex with the remaining two pebbles. If two pebbles are on the same vertex, then one pebble can be moved to the target vertex. If two pebbles are distributed on two different vertices, then

1. The placement of $n + 2t - 5$ pebbles must have been placed on one of these vertices, and one pebble can be moved to the target vertex.
2. If $n + 2t - 5$ pebbles are not placed on both vertices, then $n + 2t - 5$ pebbles are placed on the $n - 3$ vertices, so t pebbles can easily be moved to the target vertex.

Thus, $\lambda_t(K_n) = n + 2t - 3$.

4. Conclusion

In this paper we determine the t-CF pebbling number of some standard graphs such as path, star, wheel and fan graphs. The t-CF pebbling number of other standard graphs is an open problem.

References

[1] Chung, F. R. K. "Pebbling in Hypercubes" *SIAM Journal on Discrete Mathematics*, vol. 2, 1989, pp. 467–472.

[2] Muthulakshmi@Sasikala, C., Nithya, S., and Arul Steffi, A. "CF Pebbling Number of Path and Path Related Graphs" *Journal of Computational Analysis and Applications*, vol. 33, no. 8, 2024, pp. 547–552.

[3] Yamini M. Parmar, "Edge Vertex Prime Labeling for Wheel, Fan and Friendship Graph," *International Journal of Mathematics and Statistics Invention (IJMSI)*, vol. 5, no. 8, 2017, pp. 23–29.